A METHOD OF SOLVING THE HEAT-CONDUCTION PROBLEM FOR SIMPLE LAMINATED BODIES

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The temperature is expressed as the sum of integrals of a particular form. Expressions for the required time functions are determined in accordance with prescribed boundary and contact conditions by means of the Laplace transformation. Sample solutions are given.

1. The one-dimensional heat-conduction problem for an n-layered, unbounded plane wall, unbounded cylinder, and sphere with constant thermophysical coefficients in each layer can be written in the following way:

$$\frac{\partial t_i}{\partial \tau} = a_i\left(\frac{\partial^2 t_i}{\partial r^2} + \frac{\omega}{r}\,\frac{\partial t_i}{\partial r}\right) + g_i(r,\ \tau), \qquad (1)$$

where $i = 1, 2, \ldots, n$; $R_{i-1} < r < R_i$; R_{i-1}, R_i are the coordinates of the boundary surfaces of the layer (Fig. 1); $t_i(r, \tau)$ is the temperature of the i-th layer at point r at instant τ; a_i is the thermal conductivity of the i-th layer; $g_i(r, \tau)$ are prescribed functions; $\omega = 0, 1, 2$, respectively, for a plate, cylinder, and sphere. The initial and boundary conditions are:

$$t_i(r,\ 0) = f_i(r), \qquad (2)$$

$$\alpha_0 \frac{\partial t_1(R_0,\ \tau)}{\partial r} + \beta_0 t_1(R_0,\ \tau) = \varphi_0(\tau), \qquad (3)$$

$$\alpha_n \frac{\partial t_n(R_n,\ \tau)}{\partial r} + \beta_n t_n(R_n,\ \tau) = \varphi_n(\tau), \qquad (4)$$

α_0, α_n, β_0, β_n are numbers, $\varphi_0(\tau)$ and $\varphi_n(\tau)$ are prescribed functions of time. On the contact surfaces be-

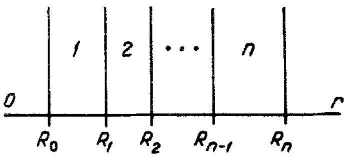

Fig. 1. n-layered wall.

tween the layers, the following conditions are prescribed:

$$t_i(R_i,\ \tau) = t_{i+1}(R_i,\ \tau),\quad i = 1,\ 2,\ \ldots,\ n-1, \qquad (5)$$

$$\lambda_i \frac{\partial t_i(R_i,\ \tau)}{\partial r} = \lambda_{i+1} \frac{\partial t_{i+1}(R_i,\ \tau)}{\partial r}; \qquad (6)$$

λ_i is the thermal conductivity in the i-th layer. We seek the solution in the form

$$t_i(r,\ \tau) = F_i + G_i + H_i + J_i, \qquad (7)$$

where

$$F_i = \int_{R_{i-1}}^{R_i} \xi^\nu f_i(\xi)\, K_i(r,\ \tau,\ \xi)\, d\xi;$$

$$G_i = \int_0^\tau d\theta \int_{R_{i-1}}^{R_i} \xi^\nu g_i(\xi,\ \theta)\, K_i(r,\ \tau-\theta,\ \xi)\, d\xi;$$

$$H_i = \int_0^\tau \chi_i(\theta)\, K_i(r,\ \tau-\theta,\ R_{i-1})\, d\theta;$$

$$J_i = \int_0^\tau \psi_i(\theta)\, K_i(r,\ \tau-\theta,\ R_i)\, d\theta.$$

For a plate $\nu = 0$, and for a cylinder and sphere $\nu = 1$. In expression (7) we have omitted the subscript ω.

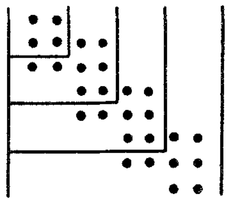

Fig. 2. Scheme for Eq. (29).

Where necessary, the text will indicate whether the system (7) refers to a plate, cylinder, or sphere. $K_i(r, \tau, \xi)$ are prescribed functions satisfying the homogeneous heat-conduction equation, i.e., when in (1) $g_i(r, \tau) = 0$, F_i, H_i, J_i are integrals of known form satisfying the homogeneous heat-conduction equation; G_i satisfies Eq. (1). The integral F_i satisfies the initial condition (2), and integrals G_i, H_i, and J_i satisfy the zero initial condition. With the stated conditions expression (7) will be a solution of Eq. (1) with the initial condition (2). To solve the problem with 2n boundary and contact conditions (3)–(6), we must determine n sums $H_i + J_i$ containing the integrand of 2n unknown functions $\chi_i(\tau)$ and $\psi_i(\tau)$, for which we put expression (7) in (3)–(6). We obtain the sytem of equations

$$\begin{aligned}
&\alpha_0 (H'_{1,0} + J'_{1,0}) + \beta_0 (H_{1,0} + J_{1,0}) &&= A_1,\\
&H_{1,1} + J_{1,1} - H_{2,1} - J_{2,1} &&= A_2,\\
&H'_{1,1} + J'_{1,1} - \frac{\lambda_2}{\lambda_1}(H'_{2,1} + J'_{2,1}) &&= A_3,\\
&\bullet \quad \bullet \quad \bullet \quad \bullet &&= \bullet\\
&\bullet \quad \bullet \quad \bullet \quad \bullet &&= \bullet\\
&H_{n-1,\,n-1} + J_{n-1,\,n-1} - H_{n,\,n-1} - J_{n,\,n-1} &&= A_{2n-2},\\
&H'_{n-1,\,n-1} + J'_{n-1,\,n-1} - \frac{\lambda_n}{\lambda_{n-1}}(H'_{n,\,n-1} + J'_{n,\,n-1}) &&= A_{2n-1},\\
&\alpha_n (H'_{n,n} + J'_{n,n}) + \beta_n (H_{n,\,n} + J_{n,\,n}) &&= A_{2n},
\end{aligned} \qquad (8)$$

where

$$A_1 = \varphi_0(\tau) - \alpha_0(F_{1/0} + G_{1/0}) - \beta_0(F_{1,0} + G_{1,0})$$

for $k = 1, 2, \ldots, n-1$;

$$A_{2k} = F_{k+1,\,k} + G_{k+1,\,k} - F_{k,\,k} - G_{k,\,k};$$

$$A_{2k+1} = \frac{\lambda_{k+1}}{\lambda_k}(F_{k+1/k} + G_{k+1/k}) - F_{k/k} - G_{k/k};$$

$$A_{2n} = \varphi_n(\tau) - \alpha_n(F_{n/n} + G_{n/n}) - \beta_n(F_{n,\,n} + G_{n,\,n}).$$

The expressions of system (8) contain the following symbols. The values of the integrals F_i, G_i, H_i, and J_i at $r = R_k$ are denoted by $F_{i,\,k}$, etc.; the values of $\partial/\partial r\ F_i$, G_i, H_i, J_i at $r = R_k$ are denoted by $F_{i/k}$, and so on. We denote the unknown functions $\chi_i(\tau)$ by x_{2i-1}, $\psi_i(\tau)$ by x_{2i}, functions $K_i(r, \tau, \xi)$ by $K_{ir\xi}$, their expressions at $r = R_k$ and $\xi = R_m$ by K_{ikm}, and the values of $\partial/\partial r\ K_{ir\xi}$ by $M_{ir\xi}$. After applying the Laplace transformation $F(s) = \int_0^\infty f(\tau)\exp(-s\tau)\,d\tau$ to the originals, the left and right sides of Eqs. (8), and using the convolution theorem we obtain a system of equations for the unknown functions $\bar{x}_k$. The bar above the letter denotes the Laplace image of the corresponding functions. The expanded matrix of this system has the form

$$\left\|\begin{array}{cccccccc|c}
b_{11} & b_{12} & 0 & 0 & 0 & 0 & 0 & 0 & \bar{A}_1 \\
\bar{K}_{110} & \bar{K}_{111} & -\bar{K}_{211} & -\bar{K}_{212} & 0 & 0 & 0 & 0 & \bar{A}_2 \\
\bar{M}_{110} & \bar{M}_{111} & -\frac{\lambda_2}{\lambda_1}\bar{M}_{211} & -\frac{\lambda_2}{\lambda_1}\bar{M}_{212} & 0 & 0 & 0 & 0 & \bar{A}_3 \\
0 & 0 & \bullet & \bullet & \bullet & \bullet & 0 & 0 & \bullet \\
0 & 0 & \bullet & \bullet & \bullet & \bullet & 0 & 0 & \bullet \\
0 & 0 & 0 & 0 & \bar{K}_{n-1,\,n-1,\,n-2} & \bar{K}_{n-1,\,n-1,\,n-1} & -\bar{K}_{n,\,n-1,\,n-1} & -\bar{K}_{n,\,n-1,\,n} & \bar{A}_{2n-2} \\
0 & 0 & 0 & 0 & \bar{M}_{n-1,\,n-1,\,n-2} & \bar{M}_{n-1,\,n-1,\,n-1} & -\frac{\lambda_n}{\lambda_{n-1}}\bar{M}_{n,\,n-1,\,n-1} & -\frac{\lambda_n}{\lambda_{n-1}}\bar{M}_{n,\,n-1,\,n} & \bar{A}_{2n-1} \\
0 & 0 & 0 & 0 & 0 & 0 & b_{2n,\,2n-1} & b_{2n,\,2n} & \bar{A}_{2n}
\end{array}\right\| \quad (9)$$

The first and last lines of the unexpanded matrix contain two, and the rest four, nonzero terms. $b_{11} = \alpha_0\bar{M}_{100} + \beta_0\bar{K}_{100}$, $b_{12} = \alpha_0\bar{M}_{101} + \beta_0\bar{K}_{101}$, $b_{2n,\,2n-1} = \alpha_n\bar{M}_{n,\,n,\,n-1} + \beta_n\bar{K}_{n,\,n,\,n-1}$, $b_{2n,\,2n} = \alpha_n\bar{M}_{n,\,n,\,n} + \beta_n\bar{K}_{n,\,n,\,n}$. The solution of the nonhomogeneous system for the images can be obtained from the Cramer formulas

$$\bar{x}_j = d_j : d,\quad j = 1, 2, \ldots, n, \qquad (10)$$

where d is a determinant composed of terms of the unexpanded matrix (9); d_j is a determinant obtained by replacement of the j-th column in the determinant d by a column of free terms of the system for the images. Since $\bar{H}_i = \bar{K}_{ir,\,i-1}\bar{x}_{2i-1}$ and $\bar{J}_i = \bar{K}_{iri}\bar{x}_{2i}$, then, reversing the Laplace transformation, we obtain

$$H_j + J_j = \frac{1}{2\pi i}\times$$

$$\times\int_{\sigma-i\infty}^{\sigma+i\infty}(\bar{K}_{jr,\,j-1}\bar{x}_{2j-1} + \bar{K}_{jrj}\bar{x}_{2j})\exp s\tau\,ds, \qquad (11)$$

where s is a variable in the complex region; σ is the value on the real axis forming the left boundary of the region of regularity of the integrand.

2. We can select the following functions $K_i(r, \tau, \xi)$ which satisfy the above conditions for the integrals from (7).

For a plate

$$K_i(r, \tau, \xi) = \frac{1}{2\sqrt{a_i\pi\tau}}\exp\left\{-\frac{(r-\xi)^2}{4a_i\tau}\right\}. \qquad (12)$$

From (12)

$$M_i(r, \tau, \xi) = -\frac{r-\xi}{4\sqrt{\pi}\,a_i^{1.5}\tau^{1.5}}\exp\left\{-\frac{(r-\xi)^2}{4a_i\tau}\right\}. \qquad (13)$$

According to [1]

$$\bar{K}_i(r, \tau, \xi) = \frac{1}{2\sqrt{a_i s}}\exp\left\{-\frac{|r-\xi|}{\sqrt{a_i}}\sqrt{s}\right\}. \qquad (14)$$

When $r < \xi$

$$\bar{M}_i(r, \tau, \xi) = \frac{1}{2a_i}\exp\left\{-\frac{\xi-r}{\sqrt{a_i}}\sqrt{s}\right\}. \qquad (15)$$

when $r > \xi$

$$\overline{M}_i(r, \tau, \xi) = -\frac{1}{2a_i}\exp\left\{-\frac{r-\xi}{\sqrt{a_i}}\sqrt{s}\right\}. \quad (16)$$

The limiting values of $\overline{M}_i$ when $r \to \xi$ are obtained from formulas (15) and (16) with $r = \xi$.

For a sphere:

$$K_i(r, \tau, \xi) = \frac{1}{2\sqrt{a_i\pi\tau}\,r}\exp\left\{-\frac{(r-\xi)^2}{4a_i\tau}\right\}, \quad (17)$$

$$M_i(r, \tau, \xi) = \frac{1}{r}\{D - K_i(r, \tau, \xi)\}, \quad (18)$$

$$D = -\frac{r-\xi}{4\sqrt{\pi}\,a_i^{1.5}\tau^{1.5}}\exp\left\{-\frac{(r-\xi)^2}{4a_i\tau}\right\},$$

$$\overline{K}_i(r, \tau, \xi) = \frac{1}{2\sqrt{a_i s}\,r}\exp\left\{-\frac{|r-\xi|}{\sqrt{a_i}}\sqrt{s}\right\}, \quad (19)$$

$$\overline{M}_i(r, \tau, \xi) = \frac{1}{r}\{\overline{D} - \overline{K}_i(r, \tau, \xi)\}. \quad (20)$$

The value of D is the same as the value of $M_i(r, \tau, \xi)$ for a plate from formula (13) and, hence, the value of $\overline{D}$ is determined from formulas (15) and (16). If we use function (17) and $R_0 = 0$, then condition (3) is replaced by

$$\frac{\partial t_1(0, \tau)}{\partial r} = 0. \quad (21)$$

Here $\alpha_0 = 1$, $\beta_0 = 0$, and the terms of the first line of the matrix (9) can be replaced by the following:

$$\bar{A}_1 = -\lim_{r\to 0} r^2\{-(\overline{F}_{1\,0} + \overline{G}_{1\,0})\},$$

$$b_{11} = -\lim_{r\to 0} r^2\overline{M}_{100} = \frac{1}{2\sqrt{a_1 s}},$$

$$b_{12} = -\lim_{r\to 0} r^2\overline{M}_{101} = \frac{1}{2\sqrt{a_1 s}}\exp-\frac{R_1\sqrt{s}}{\sqrt{a_1}}.$$

If $R_0 = 0$, then, assuming solution (7) continuous in the region bounded by the surface $r = R_1$, condition (3) can be eliminated and the problem solved for $2n - 1$ contact and boundary conditions and $2n - 1$ unknown functions. In the first layer we assume $\chi_1 = H_1 = 0$. In system (8) the first line and $H_{1,1}$ and H_1/l_1 in the second and third lines are omitted. Matrix (9) is accordingly reduced to the first line and the first column. In the case of the reduced system the finiteness of $t_1(0, \tau)$ can be ensured by the function

$$K_1(r, \tau, \xi) = \frac{\exp\left\{-\frac{(r-\xi)^2}{4a_1\tau}\right\} - \exp\left\{-\frac{(r+\xi)^2}{4a_1\tau}\right\}}{2\sqrt{a_1\pi\tau}\,r}; \quad (22)$$

function (17) is not suitable.

For a cylinder

$$K_i(r, \tau, \xi) = \frac{1}{2a_i\tau}\exp\left\{-\frac{r^2+\xi^2}{4a_i\tau}\right\} I_0\left(\frac{r\xi}{2a_i\tau}\right), \quad (23)$$

$$M_i(r, \tau, \xi) = \frac{1}{4a_i^2\tau^2}\exp\left\{-\frac{r^2+\xi^2}{4a_i\tau}\right\} \times$$

$$\times\left\{\xi I_1\left(\frac{r\xi}{2a_i\tau}\right) - rI_0\left(\frac{r\xi}{2a_i\tau}\right)\right\}. \quad (24)$$

Here and below $I_0(z)$, $I_1(z)$, $K_0(z)$ are modified Bessel functions of the first and second kind, respectively, and of order indicated by the subscript. When $r > \xi > 0$, $(r + \xi)/2(a_i)^{1/2} > (r - \xi)/2(a_i)^{1/2} > 0$, and, hence, from [2]

$$\overline{K}_i(r, \tau, \xi) = \frac{1}{a_i}K_0\left(\frac{r\sqrt{s}}{\sqrt{a_i}}\right)I_0\left(\frac{\xi\sqrt{s}}{\sqrt{a_i}}\right). \quad (25)$$

When $\xi > r > 0$

$$\overline{K}_i(r, \tau, \xi) = \frac{1}{a_i}K_0\left(\frac{\xi\sqrt{s}}{\sqrt{a_i}}\right)I_0\left(\frac{r\sqrt{s}}{\sqrt{a_i}}\right). \quad (26)$$

Assuming that it is permissible to alter the order of integration with respect to τ and differentiation with respect to r, we can write $\overline{M}_i(r, \tau, \xi) = \partial/\partial r\,\overline{K}_i(r, \tau, \xi)$. Hence, when $r > \xi > 0$

$$\overline{M}_i(r, \tau, \xi) = -\frac{\sqrt{s}}{a_i^{1.5}}K_1\left(\frac{r\sqrt{s}}{\sqrt{a_i}}\right)I_0\left(\frac{\xi\sqrt{s}}{\sqrt{a_i}}\right), \quad (27)$$

when $\xi > r > 0$

$$\overline{M}_i(r, \tau, \xi) = \frac{\sqrt{s}}{a_i^{1.5}}K_0\left(\frac{\xi\sqrt{s}}{\sqrt{a_i}}\right)I_1\left(\frac{r\sqrt{s}}{\sqrt{a_i}}\right). \quad (28)$$

The limiting values of functions $\overline{K}_i$ and $\overline{M}_i$ when $r \to \xi$ are obtained directly from formulas (25)–(28). The formula for $\overline{K}_i(\xi, \tau, \xi)$ is given in [2]. We note that differentiation with respect to the parameter ξ of the two sides of the equation

$$\int_0^\infty K_i(\xi, \tau, \xi)\exp(-s\tau)\,d\tau =$$

$$= \frac{1}{a_i}K_0\left(\frac{\xi\sqrt{s}}{\sqrt{a_i}}\right)I_0\left(\frac{\xi\sqrt{s}}{\sqrt{a_i}}\right)$$

gives a value of $\overline{M}_i(\xi, \tau, \xi)$ equal to the half-sum of the values from formulas (27) and (28) when $r \to \xi$. Function (23) ensures the finiteness of $t_1(0, \tau)$. Hence, if $R_0 = 0$, it is possible and convenient to eliminate the boundary condition (3) and to use the reduced systems (8) and (9) with changes similar to those indicated for the sphere.

3. Functions (12), (17), and (23) allow the use of operational calculus theorems, which facilitate the determination of the originals (11) in many cases. Using the Laplace theorem, we can derive the recurrent formula, by which we can represent the determinant of the unexpanded matrix (9), i.e., the determinant of the system for the images $d = |d_{2n}|$, by a second-order determinant. From the scheme in Fig. 2, in which the nonzero terms are denoted by black dots, we can obtain:

$$|d_{2n}| = A - B, \qquad n = 2, 3, \ldots, \quad (29)$$

where

$$A = |d_{2(n-1)}|\begin{vmatrix} b_{2n-1,\,2n-1} & b_{2n-1,\,2n} \\ b_{2n,\,2n-1} & b_{2n,\,2n} \end{vmatrix};$$

$$B = \left|\boxed{d_{2(n-1)}}\right|\begin{vmatrix} b_{2n-2,\,2n-1} & b_{2n-2,\,2n} \\ b_{2n,\,2n-1} & b_{2n,\,2n} \end{vmatrix}.$$

The determinant $\boxed{d_{2(n-1)}}$ is formed from $|d_{2(n-1)}|$ if in the latter the terms of the line of numbers 2(n − 1) are replaced by terms of the line of numbers 2(n − 1) + + 1 from the same columns. The analog of formula (29) for the case where the first lines and columns in matrix (9) are reduced is obtained from the scheme in Fig. 2 with the same reductions. The subscripts, numbers of lines, and columns are reduced by unity.

We can show that as a function of variable s, the determinant d ≠ 0. Formulas (10) give the only values for the images $\bar{x}_j$. In general, the considered problem has a single solution, since for the difference of two solutions of Eq. (1), obtained for the same conditions, functions g_i and the right sides of conditions (2)–(4) are zero. In this case the transformed system (8) will give all $\bar{x}_j = 0$. Hence, the difference of the solutions will be zero.

Example No. 1. Data: $n = 1$, $R_0 = t_1(r, 0) = g_1(r, \tau) = \partial t_1(0, \tau)/\partial r = 0$, $\partial t_1(R_1, \tau)/\partial r = q = \text{const}$.

Solution. a) $\omega = 0.2$. It follows from the prescribed conditions that $\alpha_0 = \alpha_1 = 1$, $\beta_0 = \beta_1 = 0$.

System (8)

$$H_{1/0} + J_{1/0} = 0$$
$$H_{1/1} + J_{1/1} = q$$

Matrix (9)

$$\left\| \begin{array}{cc|c} \bar{M}_{100} & \bar{M}_{101} & 0 \\ \bar{M}_{110} & \bar{M}_{111} & \bar{q} \end{array} \right\|$$

$$\bar{\chi}_1 = -\frac{\bar{q}\,\bar{M}_{101}}{|d|}, \quad \bar{\psi}_1 = \frac{\bar{q}\,\bar{M}_{100}}{|d|},$$

$$\bar{H}_1 + \bar{J}_1 = \bar{\chi}_1 \bar{K}_{1r0} + \bar{\psi}_1 \bar{K}_{1r1} = \bar{q}\,\frac{\bar{M}_{100}\bar{K}_{1r1} - \bar{M}_{101}\bar{K}_{1r0}}{\bar{M}_{100}\bar{M}_{111} - \bar{M}_{110}\bar{M}_{101}}.$$

Substitution of functions $\bar{K}_i$ and $\bar{M}_i$ from formulas (14)–(16), (19), (20) in the last expression gives on the right the images for a plate and sphere, agreeing with those obtained in [1] for the same problem. Hence, we can use the inversion of these images from [1].

b) $\omega = 1$, $\alpha_0 = \alpha_1 = 1$, $\beta_0 = \beta_1 = 0$. The reduced system (8): $J_{1/1} = q$. The reduced matrix (9): $\|\bar{M}_{111}\,|\,\bar{q}\|$. $\bar{\psi}_1 = \bar{q} : \bar{M}_{111}$, $\bar{J}_1 = \bar{\psi}_1\bar{K}_{1r1} = \bar{q}\bar{K}_{1r1}/\bar{M}_{111}$. Substitution of functions $\bar{K}_i$ and $\bar{M}_i$ from formulas (26) and (28) in the last expression after simple transformations gives an image corresponding to that obtained in [1] for the same problem. Hence, we can use the inversion obtained there.

Example No. 2. Data: $\omega = 0$, $n = 2$, $t_{1,2}(r, 0) = g_{1,2}(r, \tau) = R_0 = t_2(R_2, \tau) = 0$, $t_1(0, \tau) = t_c = \text{const}$.

Solution. It follows from the prescribed conditions that: $\alpha_0 = \alpha_2 = 0$, $\beta_0 = \beta_2 = 1$.

System (8)

$$H_{1,0} + J_{1,0} = t_c$$
$$H_{1,1} + J_{1,1} - H_{2,1} - J_{2,1} = 0$$
$$H_{1/1} + J_{1/1} - \frac{\lambda_2}{\lambda_1}(H_{2/1} + J_{2/1}) = 0$$
$$H_{2,2} + J_{2,2} = 0$$

Matrix (9)

$$\left\| \begin{array}{cccc|c} 1 & e_1 & & & \frac{2\sqrt{a_1}\,t_c}{\sqrt{s}} \\ e_1 & 1 & -q & -qe_2 & 0 \\ -e_1 & 1 & m & -me_2 & 0 \\ & & e_2 & 1 & 0 \end{array} \right\|$$

Matrix (9) is given after multiplication of the first and second lines by $2(a_1 s)^{1/2}$, the third line by $2a_1$, and the fourth by $2(a_2 s)^{1/2}$:

$$e_1 = \exp -\frac{R_1\sqrt{s}}{\sqrt{a_1}}, \quad e_2 = \exp -\frac{(R_2 - R_1)\sqrt{s}}{\sqrt{a_2}},$$

$$q = \sqrt{\frac{a_1}{a_2}}, \quad m = \frac{\lambda_2 a_1}{\lambda_1 a_2}.$$

The solution for the images is

$$\bar{t}_1(r, s) = \bar{H}_1 + \bar{J}_1 = \frac{d_1\bar{K}_{1r0} + d_2\bar{K}_{1r1}}{d},$$

$$\bar{t}_2(r, s) = \bar{H}_2 + \bar{J}_2 = \frac{d_3\bar{K}_{2r1} + d_4\bar{K}_{2r2}}{d}.$$

Substitution in the last expression of the expanded values for d and d_j and the values of $\bar{K}_{ir\xi}$ from formula (14), and the reduction of the exponential function to a hyperbolic one after matching the employed coordinate systems gives expressions similar to those obtained in [1] for the same problem. Hence, we can use the formulas given there for the inversions from $\bar{t}_1$ and $\bar{t}_2$.

REFERENCES

1. A. V. Luikov, Heat Conduction Theory [in Russian], GITTL, 1952.

2. V. A. Litkin and A. P. Prudnikov, Handbook of Operational Calculus [in Russian], Izd. Vysshaya shkola, 1965.

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